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NATURAL VIBRATIONS AND INSTABILITY OF PLANE FRAMES: EXACT ANALITYCAL SOLUTIONS USING POWER SERIES

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Abstract:	The objective of this article is to introduce a practical procedure for determining analytical solutions to free vibration and instability problems related to plane frames, by means of extended power series method. Transfer conditions are applied in order to guarantee geometric continuity and simultaneous equilibrium of knots or conexions. This procedure leads to an important reduction in the number of unknowns to be handled. In the problem of eigenvalue calculation of a frame (both in dynamics or statics), the solution corresponds to the nullity of a determinant whose order is substantially smaller compared to the one found by other ways (e.g. finite element method). In order to attain better presición, other procedures require an increase in the quantity of unknowns, however in the case of power series, only the degree of power is increased without enlarging the number of unknowns. A number of examples are presented in order to show the advantages of the present procedure. Moreover comparisons of computational costs are included in the examples.

September 7, 2020

Editor-in-Chief
Applied Mathematical Modelling
Prof. Johann Sienz

Dear Editor:

I am pleased to submit an original research article entitled “Natural vibrations and instability of plane frames: exact analytical solutions using power series” by Héctor Martín, Claudio Maggi, Marcelo Piovan, Anna De Rosa and Nicolás Martin Gutbrod for possible publication in *pplied Mathematical Modelling.*

In this manuscript, we show a calculation methodology of buckling and natural frequencies of vibration applied to plane frames. The method consists in using power series for modeling displacements, strains and stress-resultants of each bar of the frame. The highlight of this approach resides in a strong reduction in the number of actual unknowns, and consequently of computational processing times. Unlike other methods, the quantity of unknowns is independent of the order of the polynomial used for its representation in power series expansion.
On behalf of all authors, I believe that the present manuscript matches the aims and scopes of *Applied Mathematical Modelling*
The manuscript has not been published nor is it being reviewed in other journals.

Thank you for your consideration

Sincerely yours,

Marcelo Tulio Piovan
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HIGHLIGHTS

- **Application of a particular procedure of generalized power series method for exact determination of natural frequencies of vibration and buckling load of complex planer frames.**

- **Employement of the theory of second order in frames members with operational simplicity and high reduction of computational cost.**

- **Implementations of Iterative algorithm of power series leading to High accuracy (eventually exact) responses together with a reduced number of unknowns.**

NATURAL VIBRATIONS AND INSTABILITY OF PLANE FRAMES: EXACT ANALITYCAL SOLUTIONS USING POWER SERIES

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ABSTRACT

The objective of this article is to introduce a practical procedure for determining analytical solutions to free vibration and instability problems related to plane frames, by means of extended power series method. Transfer conditions are applied in order to guarantee geometric continuity and simultaneous equilibrium of knots or conexions. This procedure leads to an important reduction in the number of unknowns to be handled. In the problem of eigenvalue calculation of a frame (both in dynamics or statics), the solution corresponds to the nullity of a determinant whose order is substantially smaller compared to the one found by other ways (e.g. finite element method). In order to attain better presición, other procedures require an increase in the quantity of unknowns, however in the case of power series, only the degree of power is increased without enlarging the number of unknowns. A number of examples are presented in order to show the advantages of the present procedure. Moreover comparisons of computational costs are included in the examples.

Key words: natural vibrations, power series, second order theory, plane frames.

INTRODUCTION

In structural mechanics, it is a common practice to apply approximate methods of superior analysis, such as variational approaches, the mesh method and the power series among other. Although the Finite Element Method (FEM), among others, is widely and successfully employed in the numerical calculation of structural problems, it has to be recognized that the Method of Power Series (MPS) may cover a more general context [1-3], because its utilization in structural problems overcomes many difficulties of mathematical models of structures, particularly in the case of planar frames with complex boundary conditions. Moreover, the MPS has been used for a long time in the resolution of complex systems of differential equations as well.

In recent years several investigations have been presented in the calculation of plane frames using other methods than FEM or MPS. Rezaiee-Pajand et al. [4] studied the problem of free vibrations in plane frames, by applying the differential transformation method (DTM). These authors paid special attention to the derivation of governing differential equations together with the boundary conditions and the compatibility of the problem. Lee [5] analyzed the vibration performance of simple plane frame modeled with beams and columns. Typical examples of this kind of frames are often found in many architectural structures as well as industrial support facilities. In the dynamic analysis, the axial and longitudinal displacements of the beams are taken into account. Mei [6] derived an analytical solution using a wave-based method in order to study vibrations of plane frames. Under this criterion, the motion is described as waves propagating along a uniform structural element; so the waves are reflected and transmitted through discontinuities, such as structural joints. Ma [7], on the basis of exact solutions for axial vibration of elastic bars, evaluated the dynamics of elastic compex frame structures. Some of these authors managed to find exact solutions for free vibration and harmonic analysis in non-trivial cases. Moreover they introduced new beam elements by enhancing shape functions for the coupled transverse displacement field using the solutions of the homogeneous governing equations.

Tsai [8] proposed that, in the exact dynamic analysis of the flat plane frames, the effect of the distribution of the mass in the beam elements must be considered. This is achieved using the method of dynamic rigidity. Galvao et al. [9] developed a finite element scheme for the non-linear analysis of buckling and vibration of thin elastic structures with semirigid connections. These authors paid special attention to the influence of static preload on natural frequencies and modal shapes, non-linear frequency-amplitude relationships and resonance curves. Moreover, they analyzed structural systems with important practical applications: an L-shaped plane frame, shallow archs and a plane frame with inclined roof. Their results show the importance of the static preload and the rigidity of the semirigid connections in the buckling and vibration behavior of this type of structures. Rezaiee-Pajand and Khajavi [10] analyzed the vibrations in flat lattices, in which stiffness and mass matrices are optimized to obtain better performance in the eigenvalue calculation for free vibrations of plane frames. The matrices obtained are easily parameterized due to their simple structure. In this study both Bernoulli-Euler and Timoshenko beams elements were implemented. Dias and Alves [11] developed a dynamic stiffness matrix approach in order to solve the nonlinear eigenvalue problem of plane frames with end complexities modeled with Timoshenko beam theories. They calculated the natural frequencies and corresponding modal shapes for different combinations of boundary conditions and different arrangements of the bars. Mei [12] performed a free vibration analysis of flat single-story multi-bay planar frames by means of closed form solutions. The motion of the structure was described as waves propagating along uniform structural elements and reflecting and transmitting through structural discontinuities. These works took into account the coupling effects between bending and longitudinal vibrations.

Chen and Ma [13] based on the general solution for the homogeneous governing equation for the linear buckling analysis of the Euler beam, constructed new shape functions and a new finite element. Zhang et al. [14] studied the phenomenon of buckling by means of the method of quadrature elements in flat plane frames. This method begins approximating the integrands of the variational formulation of a problem. Neither the nodes nor the number of nodes in a quadrature element are fixed, so they are adjusted according to convergence requirements. It is shown that the proposed method is suitable for the buckling analysis of flat structures with variable or constant cross sections. Lee and Han [15] presented the post-buckling analysis of a semi-rigid elasto-plastic spatial plane frame with finite rotation. The structural componentes had symmetrical cross sections and semi-rigid joints. The effect of the axial forces on the bending moment and lateral buckling was considered. The Eulerian equations for a beam-column with finite rotation were taken into account. The tilting effects are adopted for an elastic system and then extended to an inelastic system with the plastic hinge concept. Non-linear buckling analyzes were performed for the spatial plane frame, in order to demonstrate the potential of the developed method in terms of precision and efficiency. Rezaiee-Pajand et al. [16] analyzed the buckling of steel frames with conical members and flexible connections. The method is based on finding the exact solutions of the governing differential equations for the stability of a plane frame with cross sections of standard I-shapes. For several particular cases, commonly used, the influences of different variables were studied, e.g. the shape factor, the conicity ratio, the lenght relation, the flexibility of the connections, the elastic restrictions of rotation and translation in the critical load, in addition to the corresponding equivalent effective length coefficient. Under other approaches [17] the buckling loads are calculated by increasing the parametrized eigenvalue incorporated in the determinant of the system matrix. In this work, the bending of plane frames and beams was modeled with finite elements, and the non-linear equilibrium equations were solved using the Newton-Raphson method. A model of finite elements for the analysis of buckling in plane frames was constructed by Sena Cardoso and Rasmussen [18]. The model was constructed on the basis of shell elements that add geometric imperfections and the semi-rigid behavior of the joints. The geometric imperfections were incorporated by projection of the buckling modal shapes in the further motion of the structure.

According to the previous context, the numerical approximations to solve the dynamic and instability problems of plane frame mechanics are extensively employed although with an important computational effort in order to attain good convergence, particularly in the presence of complexities in the structure such as non-inearities, special boundary conditions or irregular internal parts among other. These circumstances stimulate the use of methods that offer the possibility to get a closed form solution or exact solution with arbitrary precision, in the type of aforementioned structural problems. Filipich and coworkers [1-3,19,20] developed iterative and reductive procedures of the MPS applied to many structural problems with several complexities. The main focus of their contributions lays in getting affordable models and solutions for structural calculations.

In the present article, the MPS is employed to calculate the exact (or at east with arbitrary precision) eigenvalues for vibration or buckling of general plane frames. An iterative methodology is proposed to reduce the number of unknowns, tending to save computational effort. Moreover the buckling of planar frames whose members are modeled according to second order theory is also analyzed. The use of the MPS in generalized planar frames is explained and the iterative procedure for shrinking nodal unknowns is particularly detailed. Several studies are carried out in order to show the accuracy and practical effectiveness of the present methodology. Some comparisons with former finite element approaches, as well as shell finite elements of the commercial programs are performed, as well.

METHODOLOGY

POWER SERIES APPLIED TO CLASSICAL PLANE FRAMES THEORIES

The problem of natural vibrations of plane frame is linear and to solve it with computational approaches (e.g. finite elements, differential quadrature, among others), each bar or part must be divided into elements with a given polynomical approach. However, in order to reach to a greater precision, the number of subdivisions and, consequently, the number of unknowns should be increased. The use of series of powers to simulate the modal form of each section without approximations, that is, with arbitrary precision, has the advantage that it only requires static and geometric continuity in nodes of consecutive sections [1-3].

Figure 1 shows a closed planar plane frame, referred to a coordinate system of reference X-Y. The nomenclature used is the following:

nb is the total number of bars.
nn is the total number of nodes.
j is the subscript that denotes the bar, $j = 1, 2, \ldots, nb$.
n is the subscript that denotes the node, $n = 1, 2, \ldots, nn$.

The numbering of the bars and the nodes can be done arbitrarily (Figure 1), since as part of the investigation; an algorithm has been designed to re-number them conveniently.

The characteristics of each bar are the following:

E_j Young's module of the *j*-bar.
ρ_j Uniform density of *j*-bar.
F_j Area of the cross section of *j*-bar.
J_j Moment of Inertia of the *j*-bar
α_j Angle between the *j*-bar and the abscissa axis.
$\alpha_{j,k}$ Relative angle between *j*-bar and *k*-bar (Figure 2).
a_j Length of j-bar.
X-Y Global coordinates system.
x_j Local coordinate of the *j*-bar.

The units that are used are the ones that each user selects, in the examples that are shown, the MKS system has been used.

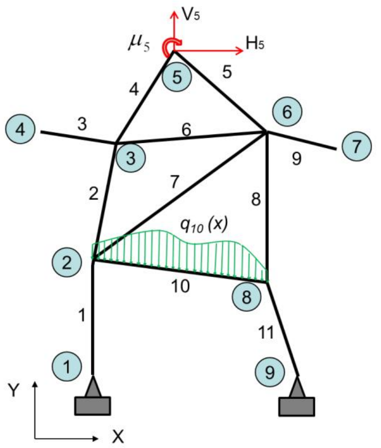


Figure. 1: General scheme of a plane frame in study.

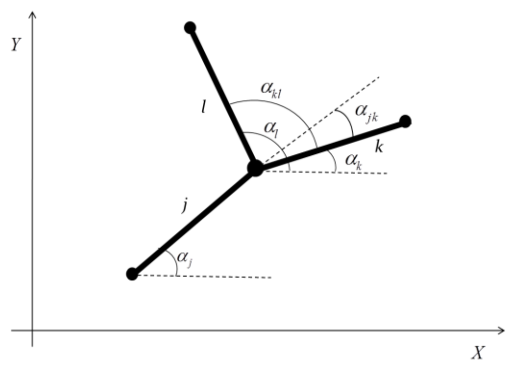


Figure 2: Location of the angles in the bars.

Energy formulation

A local coordinate system x_j is considered in each bar *j*, where each point, when the be bar is vibrating or it is loaded, will have a transverse displacement $v_j\ (x_j)$ and an axial displacement $u_j\ (x_j)$, as shown in Figure 3.

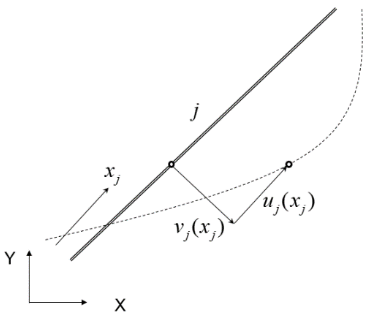


Figure 3: Displacements and local coordinates of each *j*-bar.

The potential forces of deformation U and kinetic energy K of the gantry will be the sum of those of each member. All of these depending on the displacements, that is:

$$U = \sum_{j=1}^{nb} U_j \qquad\qquad K = \sum_{j=1}^{nb} K_j \tag{1a, b}$$

In which the potential energy (2) will be depending on moments and normal stresses:

$$U_j = \frac{1}{2}\left[\int_0^{a_j} \frac{M^2(x_j)}{E_j J_j}\, dx_j + \int_0^{a_j} \frac{N^2(x_j)}{E_j F_j}\, dx_j\right] \tag{2}$$

The Kinetic Energy (3), accepting normal modes of vibration, is expressed as a function of ω (circular frequency of vibration of the frame), in addition to the transverse and axial displacements.

$$K_j = \frac{1}{2}\rho_j F_j\, \omega^2 \int_0^{a_j} \left[v^2(x_j) + u^2(x_j)\right] dx_j \tag{3}$$

The expressions corresponding to the bending moments *M* and the normal stresses *N* are indicated below:

$$M(x_j) = -E_j J_j\; v_j''\left(x_j\right) \qquad\qquad N(x_j) = E_j F_j\; u_j'\left(x_j\right) \tag{4a, b}$$

Where:

$$(\otimes)' = \frac{\partial(\otimes)}{\partial x_j} \tag{5}$$

According to the Hamilton's Theorem [17-19]:

$$\delta(U - K) = 0 \tag{6}$$

substituting expressions (1) in (6) and taking into account the expressions (2), (3) and (4a, b) results:

$$\delta\left(\frac{1}{2}\sum_{j=1}^{nb}\left\{\int_0^{a_j} \frac{\left(E_j J_j\; v_j''\left(x_j\right)\right)^2}{E_j J_j}\, dx_j + \int_0^{a_j} \frac{\left(E_j F_j\; u_j'\left(x_j\right)\right)^2}{E_j F_j}\, dx_j - \right.\right.$$
$$\left.\left. - \rho_j F_j\, \omega^2 \int_0^{a_j} \left[v_j^2(x_j) + u_j^2(x_j)\right] dx_j \right\}\right) = 0 \tag{7}$$

For arbitrary variations, in all the bars (j = 1, 2,..., nb), one arrives at the differential equations:

$$\begin{aligned} E_j J_j\, v_j''''\left(x_j\right) - \rho_j F_j\, \omega^2\, v_j(x_j) &= 0 \\ E_j F_j\; u_j''\left(x_j\right) + \rho_j F_j\, \omega^2 u_j(x_j) &= 0 \end{aligned} \tag{8a, b}$$

The equations are transformed in a non-dimentional form by means of the following change of variables:

$$0 \le x_j \le a_j \quad \Rightarrow \quad 0 \le \frac{x_j}{a_j} \le 1 \quad \Rightarrow \quad \xi_j = \frac{x_j}{a_j} \tag{9}$$

Henceforth, the apostrophe, indicated in (5), refers to the derivative with respect to the new variable:

$$(\otimes)' = \frac{\partial(\otimes)}{\partial \xi_j}$$

Then, the differential equations (8a) and (8b) are written as:

$$\frac{E_j J_j}{a_j^{\ 4}} v_j''''\left(\xi_j\right) - \rho_j F_j \,\omega^2 \, v_j(\xi_j) = 0 \tag{10}$$

$$\frac{E_j F_j}{a_j^{\ 2}} \; u_j''\left(\xi_j\right) + \rho_j F_j \,\omega^2 u_j(\xi_j) = 0 \tag{11}$$

that can be rewritten in the following non-dimensional form:

$$\boxed{v_j''''\left(\xi_j\right) - \Omega_j^2 \, v_j(\xi_j) = 0} \qquad \boxed{u_j''\left(\xi_j\right) + \left(\frac{\Omega_j}{\lambda_j}\right)^2 u_j(\xi_j) = 0} \tag{12a, b}$$

where:

$$\Omega_j^2 = \frac{\rho_j F_j}{E_j J_j} \omega^2 a_j^4 \qquad \lambda_j = \frac{a_j}{\sqrt{J_j / F_j}} \tag{13a, b}$$

Development of solutions in power series

It is proposed, for the solution of the differential equations (12a, b), a development in power series for the unknown functions of the transverse and axial displacements of each bar [4-7, 20-23] with the following expressions:

$$v_j\left(\xi_j\right) = \sum_{i=0}^{m} A_{j,i} \, \xi_j^i \qquad u_j\left(\xi_j\right) = \sum_{i=0}^{m} B_{j,i} \, \xi_j^i \tag{14a, b}$$

where, their corresponding derivatives (of order r), can be written in the following way:

$$v_j^r\left(\xi_j\right) = \sum_{i=0}^{m-r} \varphi_{r,i} \;\; A_{j,i+r} \, \xi_j^i \tag{15}$$

with:

$$\varphi_{r,i} = \frac{(i+r)!}{r!} \tag{16}$$

Returning to the differential equations that govern our problem (12a)-(12b), using the solutions proposed in series of powers:

$$\sum_{i=0}^{m-4} \varphi_{4,i} \;\; A_{j,i+4} \, \xi_j^i - \Omega_j^2 \sum_{i=0}^{m} A_{j,i} \, \xi_j^i = 0 \tag{17}$$

$$\sum_{i=0}^{m-2} \varphi_{2,i} \;\; B_{j,i+2} \, \xi_j^i \; + \left(\frac{\Omega_j}{\lambda_j}\right)^2 \sum_{i=0}^{m} B_{j,i} \, \xi_j^i \; = 0 \tag{18}$$

By equalizing the coefficients of equal power in both developments, for bar *j*, the following recurrence equations are deduced:

$$A_{j,i+4} = \frac{\Omega_j^2 \, A_{j,i}}{\varphi_{4,i}} \qquad (\, i = 0, 1, \ldots, m\text{-}4) \tag{19}$$

$$B_{j,i+2} = -\left(\frac{\Omega_j}{\lambda_j}\right)^2 \frac{B_{j,i}}{\varphi_{2,i}} \qquad (\, i = 0, 1, \ldots, m\text{-}2) \tag{20}$$

As one can see in equation (19), the coefficients $A_{j,i+4}$ are linked with the $A_{j,i}$, and in equation (20), the $B_{j,\, i\,+\,2}$ with the $B_{j,\, i}$ what results, in principle , each bar with a total of 6 unknowns, namely: $A_{j,0}$, $A_{j,1}$, $A_{j,2}$, $A_{j,3}$, $B_{j,0}$ y $B_{j,1}$.

Conditions of geometric compatibility, essential or primary

In each node where there are *m* bars, *m-1* must be considered as conditions of geometric continuity, essential or primary.

As shown in Figure 4, the relationships between displacements of a bar *k* starting from the node with another bar *j* arriving at the same node, are the following:

$$u_k(0) = u_j(1)\cos\alpha_{j,k} + v_j(1)\sin\alpha_{j,k} \tag{21}$$

$$v_k(0) = v_j(1)\cos\alpha_{j,k} - u_j(1)\sin\alpha_{j,k} \tag{22}$$

$$\frac{v'_k(0)}{a_k} = \frac{v'_j(1)}{a_j} \tag{23}$$

being the bar *j* that arrives at the node, according to the direction of circulation (and it is evaluated at its final end, $\xi_j = 1$) and k one of the bars that leaves the node (being evaluated at its beginning, $\xi_k = 0$). The lengths of the bars *j* and *k* are *aj* and *ak* respectively. The angle $\alpha_{j,k}$ is the relative between the bar j and the k. For the purpose of simplifying writing, the sine and cosine trigonometric relations are called between bars *k*, *j* as follows:

$$C_{j,k} = \cos\alpha_{j,k} \qquad S_{j,k} = \sin\alpha_{j,k}$$

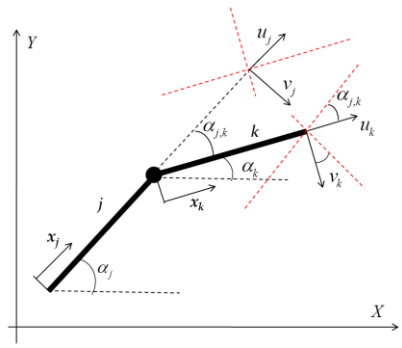


Figure 4: Scheme of displacements between two consecutive bars.

Replacing the transversal and axial displacement functions, due to their corresponding development in power series, and taking into account that, for example, for bar *j* it is:

$$v_j(0) = A_{j,0}, \quad v_j(1) = \sum_{i=0}^{m} A_{j,i}, \quad u_j(0) = B_{j,0} \quad \text{y} \quad u_j(1) = \sum_{i=0}^{m} B_{j,i}$$

The geometric conditions are expressed as follows:

$$B_{k,0} = C_{j,k}\sum_{i=0}^{m} B_{j,i} + S_{j,k}\sum_{i=0}^{m} A_{j,i} \tag{24}$$

$$A_{k,0} = C_{j,k}\sum_{i=0}^{m} A_{j,i} - S_{j,k}\sum_{i=0}^{m} B_{j,i} \tag{25}$$

$$A_{k,1} = \frac{a_k}{a_j} \sum_{i=0}^{m-1} \varphi_{1,i} \, A_{j,i+1} \tag{26}$$

Then, the coefficients $A_{k,0}$, $A_{k,1}$ and $B_{k,0}$ of the bar k are calculateed as a function of the j-bar. Taking into account the order in which the calculation process is performed, the bar j is analyzed before the bar k.

Geometric, essential or primary compatibility equations

There are situations in which two or more bars, with known coefficients, converge in a given node. For example, this is illustrated in node 8 of Figure 1, where bars 8, 10 and 11 have already been calculated if the path has started in bar 1. Under these circumnstances, the geometric compatibility equations must be considered in the bars with known coefficients in the development in series. This situation adds 3 more equations for each bar in the context described above. This procedure turns out to be one of the main advances of the present investigation since, with this methodology, it is possible to go from open frames (in which only two bars to each node concur), previously studied by other authors [4, 5, 7], to closed plane frames (in which more than two bars arrive at the nodes).

The geometric compatibility equations are expressed as follows:

$$\sum_{i=0}^{m} B_{k,i} - \left(C_{j,k} \sum_{i=0}^{m} B_{j,i} + S_{j,k} \sum_{i=0}^{m} A_{j,i} \right) = 0 \tag{27}$$

$$\sum_{i=0}^{m} A_{k,i} - \left(C_{j,k} \sum_{i=0}^{m} A_{j,i} - S_{j,k} \sum_{i=0}^{m} B_{j,i} \right) = 0 \tag{28}$$

$$a_j \sum_{i=0}^{m-1} \varphi_{1,i} A_{k,i+1} - a_k \sum_{i=0}^{m-1} \varphi_{1,i} \, A_{j,i+1} = 0 \tag{29}$$

$$a_j \; A_{k,1} - a_k \; A_{j,1} = 0 \tag{30}$$

It must be emphasized that, in these equations, the bar k is evaluated at its end: $\xi_k = 1$, resulting in a summation of the coefficients of the power series of Eq. (27)-(29). The special case of Eq. (30) is when both bars are evaluated at their origin: $\xi_k = 0$.
Therefore, with the **conditions of geometrical compatibility**, essential or primary, one can calculate coefficients of a given bar from the coefficients of bars that precede the development and that concur to the same node.

Nodal Static equilibrium or natural or secondary conditions

In each node that belongs to more tan one bar, the static equilibrium equations should be proposed, that is two sums of forces and a sum of moments (or stress resultants in other words). In the following expressions one can see, for a generic bar k, the equation of shear forces (31), the equation of bending moments (32) and the equation of normal forces (33). This generic bar k, has its origin in the node ($\xi_k = 0$). These three strees resultants are defined in terms of the nbs bars that have their origin in the same node (and evaluated as $\xi_j = 0$) and the *nbe* bars that end in the same node, and evaluated as $\xi_j = 1$.

$$Q_k(0) = \sum_{j=1}^{nbe} \left(Q_j(1) C_{kj} + N_j(1) S_{kj} \right) - \sum_{j=1}^{nbs} \left(Q_j(0) C_{kj} + N_j(0) S_{kj} \right) \tag{31}$$

$$N_k(0)=\sum_{j=1}^{nbe}\left(N_j(1)C_{kj}-Q_j(1)S_{kj}\right)+\sum_{j}^{nbs}\left(Q_j(0)S_{kj}-N_j(0)C_{kj}\right) \tag{32}$$

$$M_k(0)=\sum_{j}^{nbe}M_j(1)-\sum_{j}^{nbs}M_j(0) \tag{33}$$

Now introducing the following expressions:

$$J_{2,j}=\frac{E_jJ_j}{a_j^2} \qquad J_{3,j}=\frac{E_jJ_j}{a_j^3} \qquad F_{1,j}=\frac{E_jF_j}{a_j} \tag{34a,b,c}$$

Taking into account that:

$$N_j=F_{1,j}\,u_j'(\xi_j) \qquad Q_j=-J_{3,j}\,v_j'''(\xi_j) \qquad M_j=-J_{2,j}\,v_j''(\xi_j) \tag{35a,b,c}$$

It is posible to find the three coefficients of the bar k in terms of the coefficients of the precedeng bars:

$$\begin{aligned}A_{k,3}=\frac{-1}{J_{3,k}\,\varphi_{3,0}}\Bigg[&\sum_{b=1}^{nbe}\left(-J_{3,b}\,C_{k,b}\sum_{i=0}^{m-3}\varphi_{3,i}A_{b,i+3}+F_{1,b}\,S_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,B_{b,i+1}\right)+\\&+\sum_{b=1}^{nbs}\left(J_{3,b}\,C_{k,b}\varphi_{3,0}A_{b,3}-F_{1,b}\,S_{k,b}\varphi_{1,0}\,B_{b,1}\right)\Bigg]\end{aligned} \tag{36}$$

$$\begin{aligned}B_{k,1}=\frac{1}{F_{1,k}\,\varphi_{1,0}}\Bigg[&\sum_{b=1}^{nbe}\left(J_{3,b}\;S_{k,b}\sum_{i=0}^{m-3}\varphi_{3,i}A_{b,i+3}+F_{1,b}\,C_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,B_{b,i+1}\right)+\\&+\sum_{b=1}^{nbs}\left(-J_{3,b}\;S_{k,b}\varphi_{3,0}A_{b,3}-F_{1,b}\,C_{k,b}\varphi_{1,0}\,B_{b,1}\right)\Bigg]\end{aligned} \tag{37}$$

$$A_{k,2}=\frac{-1}{J_{2,k}\,\varphi_{2,0}}\left[\sum_{b=1}^{nbe}\left(-J_{2,b}\sum_{i=0}^{m-2}\varphi_{2,i}A_{b,i+2}\right)+\sum_{b=1}^{nbs}J_{2,b}\,\varphi_{2,0}A_{b,2}\right] \tag{38}$$

These above expressions allow the recurrence calculation for a given number of bar that concur or leaves from a given node.

Equilibrium equations, natural or secondary in the nodes

In case of arriving at a node where all the bars have already been defined, then it is the turn to work with the equilibrium equations in which all the bars' coefficients are known. This situation adds 3 more equations. The above mentioned turns out to be another of the main advances product of the present investigation since, with this methodology.

Equilibrium equations are expressed in the following way, if we consider that they are all incoming bars to the node, that is, their coefficients have already been calculated or they are unknowns of the problem.

$$\sum_{b=1}^{nbe}\left(-J_{3,b}\,C_{k,b}\sum_{i=0}^{m-3}\varphi_{3,i}A_{b,i+3}+F_{1,b}\,S_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,B_{b,i+1}\right)=0 \tag{39}$$

$$\sum_{b=1}^{nbe}\left(J_{3,b}\;S_{k,b}\sum_{i=0}^{m-3}\varphi_{3,i}A_{b,i+3}+F_{1,b}\,C_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,B_{b,i+1}\right)=0 \tag{40}$$

$$\sum_{b=1}^{nbe}\left(-J_{2,b}\sum_{i=0}^{m-2}\varphi_{2,i}A_{b,i+2}\right)=0 \tag{41}$$

It must be emphasized that, in these equations, all bars are evaluated in $\xi_j = 1$ thus obtaining a sum in the coefficients of the power series.
When an end of bar k is connected to the rest of the structure/ground by means of an articulation, it is necessary satisfy the nullity of bending moment in the node, consequently Eq. (41) can be written as follows:

$$J_{2,k}\sum_{i=0}^{m-2}\varphi_{2,i}A_{k,i+2} = 0 \tag{42}$$

THEORY OF SECOND ORDER PLANE FRAMES

When a structure with axial forces is analyzed with second-order theory, it means that equilibrium is posed in its deformed position. The problem is non-linear. The remaining hypotheses of the linear analysis are maintained: mechanical and kinematic linearity.
The problem of second order of plane frames is addressed, taking into account that structural engineering can be significantly important in flat roofs. In the metallic construction, especially for the final dimensioning, it is crucial to know the safety against the instability of the system under study, potentiality that with this theory can be approximated [21]. Second order theory should be considered as an indirect study of structural instability. In fact, when the load state, at least theoretically, approaches below to the critical state, the present theory will give rise to deformations/displacements beyond the admissible range.
The resolution of frames with the aforementioned topics is based on the use of differential governing beam equations in axial and transverse displacements, where the flexural-axial effects are coupled. However, instead of working with non-linear differential equations (since the axial stress of each bar depends on the derivative of the axial displacement and multiplies the second derivative of the transverse displacement), an iterative process is carried out, where for each step the axial stresses produced by the linearization of the process are maintened. After a few iterations, the normal stresses in successive steps converge and the solution is completed.
It should be noted that the consideration of the axial forces, within the differential equation of the flexional displacement, can lead to significant structural displacements that a first-order theory can not detect.
In parts of the structure where general concentrated forces are located, it is necessary to create a node to apply them. These forces are illustrated in point 5 of Figure 1, where: $\boldsymbol{H}_5$, $\boldsymbol{V}_5$ are horizontal and vertical applied forces and and $\boldsymbol{\mu}_5$ is the applied moment.

Governing equations of the problem

The strain energy considering bending and axial contributions for each bar *j*, is written as:

$$\boldsymbol{U} = \frac{1}{2}\left(\sum_{j=1}^{nb}\left[\int_0^{a_j}\frac{M^2\left(x_j\right)}{E_jJ_j}dx_j + \int_0^{a_j}\frac{N^2\left(x_j\right)}{E_jF_j}dx_j\right]\right) \tag{43}$$

In this deduction and for simplification purposes, it is accepted that distributed axial loads are zero. Eventually, if theses loads have to be considered, equivalent nodal forces are incorporated to the model, in the usual way.
The contribution of strain energy corresponding to the axial loads N_j in the bars (assumed constant in each iteration) and associated to second order terms, is written as:

$$\boldsymbol{G}=\frac{1}{2}\sum_{j=1}^{nb}\boldsymbol{N}_j\int_0^{a_j}\left(v_j'\left(x_j\right)\right)^2 dx_j \tag{44}$$

The energy **S** due to the forces applied in the nodes *i*.

$$\begin{aligned}\boldsymbol{S}=\sum_{i=1}^{nn}\Big\{&H_i\left[C_\alpha\, u_j\left(x_j\right)+S_\alpha\, v_j\left(x_j\right)\right]+\\&+V_i\left[S_\alpha\, u_j\left(x_j\right)-C_\alpha\, v_j\left(x_j\right)\right]+\mu_i\, v_j'\left(x_j\right)\Big\}\end{aligned} \tag{45}$$

In the previous expression, the subscript *j* refers to the bar *j* on which the stress is projected, with α being the angle between the bar and the global coordinate system, C_α and S_α are the values of cosine and sine.

The energy **T** due to transverse loads $q_j(x)$ distributed along each bar *j*, is written as:

$$\boldsymbol{T}=\sum_{j=1}^{nb}\left(\int_0^{a_j} q_j\left(x_j\right)v_j\left(x_j\right)dx_j\right) \tag{46}$$

To deduce the equilibrium equations, the following variational condition should be satisfied:

$$\delta\left(\boldsymbol{U}-\boldsymbol{G}-\boldsymbol{S}-\boldsymbol{T}\right)=0 \tag{47}$$

Substituting Eqs (43)-(40) into Eq. (47) the govering differential equations of the problem are obtained.

$$E_j J_j\, v_j''''\left(x_j\right)+\boldsymbol{N}_j\, v_j''\left(x_j\right)=q_j\left(x_j\right) \tag{48}$$

where $\boldsymbol{N}_j = E_j F_j u_j\,'(x)$

$$E_j F_j\, u_j''\ \left(x_j\right)=0 \tag{49}$$

And employing non-dimensional forms the equations are transformed in:

$$\boxed{v_j''''\left(\xi_j\right)-\frac{q_j\left(\xi_j\right)}{J_{4,j}}+\frac{\boldsymbol{N}_j\, v_j''\left(\xi_j\right)}{a_j^2\, J_{4,j}}=0} \qquad \boxed{u_j''\ \left(\xi_j\right)=0} \tag{50a,b}$$

As in previous deductions, quotes indentify derivation with respect to spatial variable:

$$\xi_j=\frac{x_j}{a_j} \qquad J_{4,j}=\frac{E_j J_j}{a_j^4} \qquad F_{2,j}=\frac{E_j F_j}{a_j^2} \tag{51a,b,c}$$

Using the power series, in the way it has been previously stated, the expressions for recurrence are obtained by equalizing the coefficients of equal power:

$$A_{j,i+4}=\frac{q_{j,i}-\dfrac{\boldsymbol{N}_j}{a_j^2}\varphi_{2,i}\ A_{j,i+2}}{J_{4,j}\,\varphi_{4,i}} \tag{52}$$

$$B_{j,i+2}=0 \tag{53}$$

In Eq (52), the coefficients $A_{j,i+4}$ are linked to the coefficients of the functions that represent the loads $q_{j,i}$, with the compression forces in each bar *Nj* and with the previous coefficients in the series development of the transverse displacement $A_{j,i+2}$. The Eq. (53) indicates that the function of the axial displacement is linear, since the coefficients in the series development greater than or equal to 2 are null.

It is worth clarifying that the conditions and geometric compatibility equations, or first order, that arise in each node have not changed either.

Conditions and static equations of equilibrium in the nodes

In the nodes to which more than one bar concur, the corresponding conditions of equilibrium should be considered, that is, two summations of stresses and one of moments, in the same way as was previously done . With a similar analysis, we arrive at the following expressions for equilibrium conditions:

$$A_{k,2}=\frac{1}{J_{2,k}\,\varphi_{2,0}}\left[\sum_{b=1}^{nbe}\left(J_{2,b}\sum_{i=0}^{m-2}\varphi_{2,i}A_{b,i+2}\right)-\sum_{b=1}^{nbs}J_{2,b}\,\varphi_{2,0}A_{b,2}-\mu_n\right] \quad (54)$$

$$\begin{aligned}A_{k,3}=\frac{1}{J_{3,k}\,\varphi_{3,0}}\Bigg[&\sum_{b=1}^{nbe}\Bigg(J_{3,b}\,C_{k,b}\sum_{i=0}^{m-3}\varphi_{3,i}A_{b,i+3}+F_{1,b}\,S_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,B_{b,i+1}+\\&+\frac{\boldsymbol{N}_b}{a_b}C_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,A_{b,i+1}\Bigg)+H_n\,S_k-V_n\,C_k+\frac{\boldsymbol{N}_k}{a_k}A_{b,k}+\\&+\sum_{b=1}^{nbs}\left(-J_{3,b}\,C_{k,b}\varphi_{3,0}A_{b,3}-F_{1,b}\,S_{k,b}\varphi_{1,0}\,B_{b,1}-\frac{\boldsymbol{N}_b}{a_b}C_{k,b}A_{b,1}\right)\Bigg]\end{aligned} \quad (55)$$

$$\begin{aligned}B_{k,1}=\frac{1}{F_{1,k}\,\varphi_{1,0}}\Bigg[&\sum_{b=1}^{nbe}\Bigg(-J_{3,b}\;\;S_{k,b}\sum_{i=0}^{m-3}\varphi_{3,i}A_{b,i+3}+F_{1,b}\,C_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,B_{b,i+1}-\\&-\frac{\boldsymbol{N}_b}{a_b}S_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,A_{b,i+1}\Bigg)-H_nC_k-V_nS_k+\\&+\sum_{b=1}^{nbs}\left(J_{3,b}\;\;S_{k,b}\varphi_{3,0}A_{b,3}-F_{1,b}\,C_{k,b}\varphi_{1,0}\,B_{b,1}+\frac{\boldsymbol{N}_b}{a_b}S_{k,b}A_{b,1}\right)\Bigg]\end{aligned} \quad (56)$$

Being, *nbe* and *nbs* the total number of incoming and outgoing bars respectively to the node *n* under study, *k* is the bar that takes values from the previous ones in the way of travel of the plane frame.

In the case in which all the bars concur to a given node, the equations of equilibrium are analized as previously, arriving to the following expressions:

$$\sum_{b=1}^{nbe}\left(J_{2,b}\sum_{i=0}^{m-2}\varphi_{2,i}A_{b,i+2}\right)-\mu_n=0 \quad (57)$$

$$\begin{aligned}&\sum_{k}^{nbe}\Bigg[-J_{3,k}C_{k,3}\sum_{i=0}^{m-3}\varphi_{3,i}A_{k,i+3}-F_{1,k}S_{k,3}\sum_{i=0}^{m-1}\varphi_{1,i}B_{k,i+1}-\\&-\frac{\boldsymbol{N}_b}{a_b}C_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,A_{b,i+1}\Bigg]-H_nS_k+V_nC_k=0\end{aligned} \quad (58)$$

$$\begin{aligned}&\sum_{k}^{nbe}\Bigg[-J_{3,k}S_{k,3}\sum_{i=0}^{m-3}\varphi_{3,i}A_{k,i+3}+F_{1,k}C_{k,3}\sum_{i=0}^{m-1}\varphi_{1,i}B_{k,i+1}-\\&-\frac{\boldsymbol{N}_b}{a_b}S_{k,b}\sum_{i=0}^{m-1}\varphi_{1,i}\,A_{b,i+1}\Bigg]-H_hC_k-V_hS_k=0\end{aligned} \quad (59)$$

For the cases in which there are generalized forces applied in a node, origin of a given bar, the expressions for calculating the variations, to the bar *k* in the node *n*, are the following:

$$A_{k,2} = \frac{-\mu_n}{J_{2,k}\ \varphi_{2,0}} \tag{60}$$

$$A_{k,3} = \frac{1}{J_{3,k}\ \varphi_{3,0}}\left(H_n\, S_k - V_n\, C_k - \frac{\boldsymbol{N}_k}{a_k} A_{b,k} \right) \tag{61}$$

$$B_{k,1} = \frac{-H_n C_k - V_n S_k}{F_{1,k}} \tag{62}$$

DISCUSSION

In order to illustrate the methodology proposed previously, some examples of plane frames are presented here, to which the first natural frequencies of vibration are calculated and in some cases their modal forms are shown. In the last examples there are plane frames where the second order theory is studied.

The results, obtained with codes programmed within the *Mathematica*® platform [22], are compared with those calculated using the commercial code ALGOR for Finite Element Analysis and with those obtained through the use of Finite Element procedures developed for arches and frames by Auciello and De Rosa [23].

Example 1

The first five natural frequencies of the plane frame of Figure 5 are calculated, which has 12 nodes and a total of 18 bars.

It is possible to choose bars with different geometric and physical characteristics, but to simplify the loading of data in the programs used, in this example all equal bars are adopted, of uniform section F=0.12 m^2, with modulus of elasticity E=2.1x10^{11} N/ m^2, the bar lengths are mesured in meters, the moment of inertia J=0.0036 m^4 and specific weight ρ=7850 Kg/m^3.

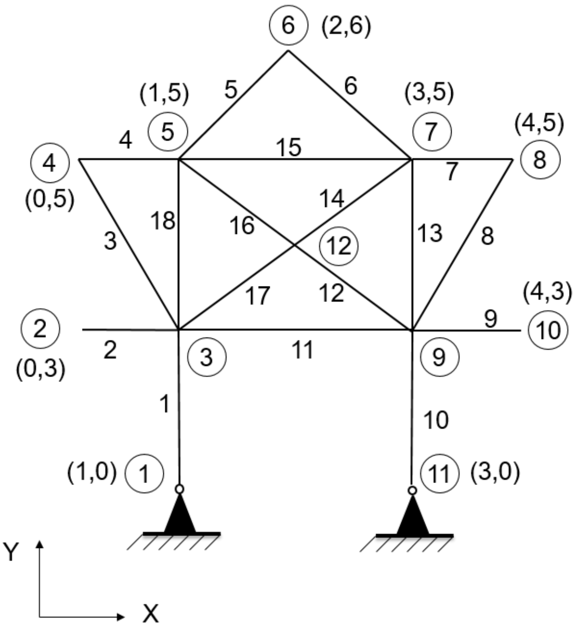


Figure 5: Plane frame analyzed with 18 bars and 12 nodes (coordinates in meters).

Using the representation in power series, this plane frame is solved with only 33 unknowns. With the method of finite elements, at 10 elements per bar, add up to 540 unknowns. The results of the first natural frequencies are shown in Table 1. In the same it can be seen that with grade 10, in power series, good results are achieved for the first three natural frequencies, requiring an increase in degree for better approximations in frequencies higher, which does not imply that the number of unknowns is increased for the resolution of this network.

Mode	Power series			FEM Algor	FEM [23]
	10	30	50		
1	69.803	69.803	69.803	69.802	69.8038
2	551.361	551.961	551.961	551.469	552.196
3	768.218	768.349	768.349	768.27	769.198
4	1259.472	1308.517	1308.517	1308.291	1310.38
5	1478.636	1506.147	1506.147	1505.922	1509.55

Table 1: First 5 frequencies [rad/s] obtained with power series of degrees 10, 30 and 50, compared with those obtained using 10 elements per bar in the Algor Software and with other FEM procedures [23].

Example 2

Asymmetrical portico extracted from the literature [10] shown in Figure 6. It has been constructed with beams (2, 4 and 7) using IPE 160 with cross-sectional area A = 20.1 cm^2, inertia I = 869 cm^4, columns (1, 3, 5, 6 and 8) with IPE 160, A = 40.2 cm^2, inertia I = 1738 cm^4, specific weight of all bars: ρ = 7850 Kg / m^3.

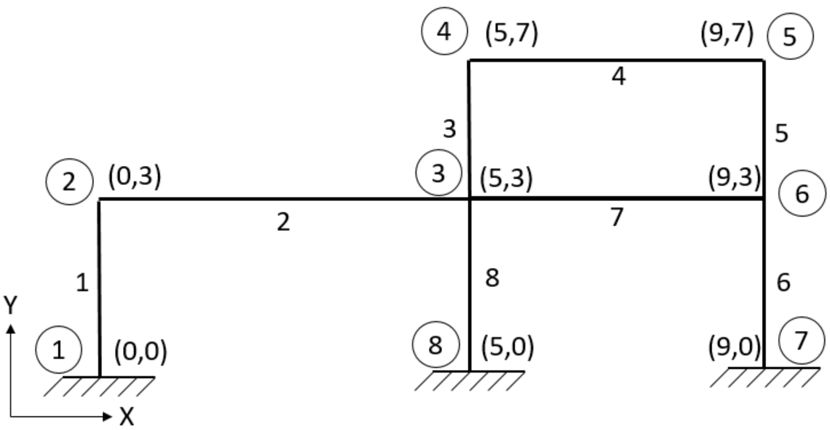
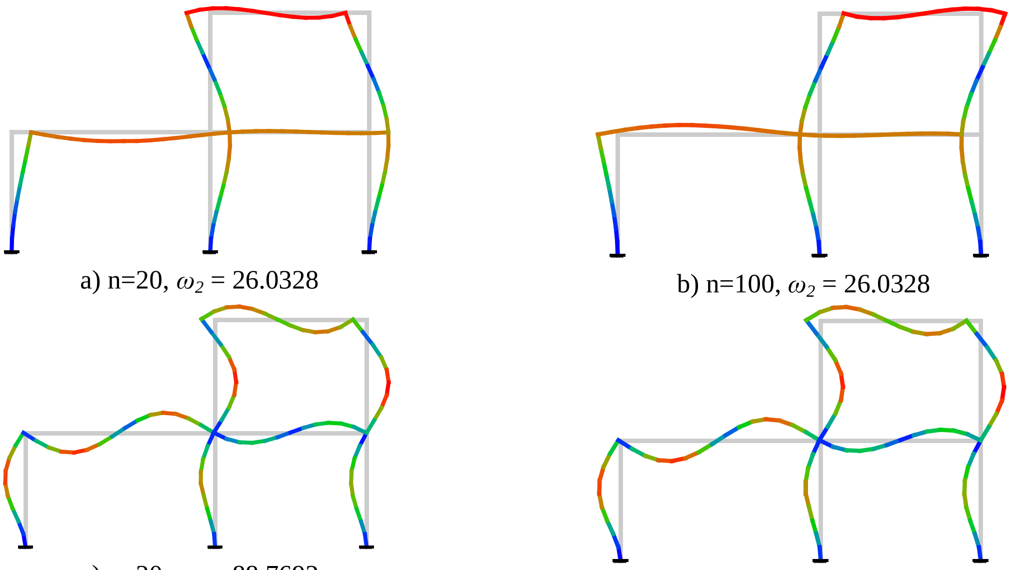


Figure 6: Frame [10] conformed by IPN 160, measure in [m].

Mode	Power series		FEM [10]	FEM [23]
	20	100		
1	8.98951	8.98951	8.9871	8.9971
2	26.0328	26.0328	26.0078	26.1736
3	40.0649	40.0649		
4	51.7715	51.7715		
5	64.1664	64.1664		

Table 2: First 5 frequencies [Hz] obtained with power series of degrees 20 and 100, compared with those obtained in reference [10] and calculated with FEM procedures [23].

It is interesting to note in this example that, the eigenvalues determined by means of power series with maximum exponent 20 have no differences with the solution employing series with maximum power 100. Consequently, it is not necessary to increase the polynomial order to get better precision. In any case, whatever the maximum exponent used, the number of unknowns in this methodology is always 12 which in fact is quite helpful to make parametric evaluations or long term repetitive calculations for example in uncertainty quantification or optimization procedures. The graphics of Figure 7 show some of the modal forms obtained with power series of degrees 20 and 100, the same modes (second and sixth just as examples) are observed with their respective natural frequencies.



a) n=20, ω_2 = 26.0328

b) n=100, ω_2 = 26.0328

c) n=20, ω_6 = 88.7692

d) n=100, ω_6 = 88.7692

Figure 7: Modal forms obtained with power series for grades 20 and 100.

It clear that, the modes shown in Figure 7 have a magnified scale for illustrative purposes.

Example 3

A closed plane frame with an important degree of complexity is resolved, which is considered one of the main contributions in this work (Figure 8). Elastic solutions are sought considering the theory of second order, using series of powers, comparing then the results with those obtained by using a finite element program [2]. The plane frame consists of 10 bars, 9 nodes, embedded in nodes 1, 7 and 8. In the diagram there are 2 point loads, P3 and P4 (in N) located in nodes 3 and 4. In addition there are 4 uniformly distributed loads q2, q3, q4 and q10 (in N / m) located in bars 2, 3, 4 and 10 respectively. The coordinates of the nodes are indicated in a global system, with units in m. The bars have the following characteristics: I = 0.0036 [m^4], E=2.1 x 10^{11} [N/m^2].

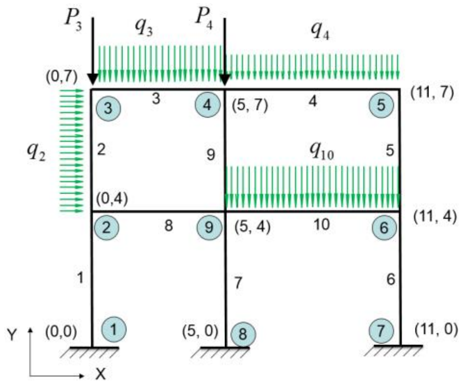


Figure 8: Plane frame of 10 bars with punctual and distributed loads.

Two load states are solved (case 1 and case 2), in each of them, increasing all the loads with a factor f, which takes values 1, 100 and 1000. The elastic solution is sought using the second order theory with Power series, using the Mathematica software, comparing the results with the finite element method. Some of the results obtained are shown in Tables 3 and 4. It has been tabulated for node 3, the horizontal displacements v_3 and vertical u_3, the rotation θ_3 and the moment M_3. The calculation time for each method used is also indicated.

Case 1: $\boldsymbol{P}_3 = 2 \times 10^5$ [N], $\boldsymbol{P}_4 = 0$, $\boldsymbol{q}_2 = 0$, $\boldsymbol{q}_3 = 0$, $\boldsymbol{q}_4 = 3 \times 10^4$ [N/m] and $\boldsymbol{q}_{10} = 0$.

		Power series	FEM
$f=1$	time [sec]	8	32
	v_3 [m]	3.048×10^{-5}	3.0×10^{-5}
	u_3 [m]	5.38×10^{-5}	5.4×10^{-5}
	θ_3 (°)	1.159×10^{-3}	1.157×10^{-3}
	M_3 [Nm]	7846.35	7835.08
$f=10^2$	time [sec]	9	74
	v_3 [m]	3.038×10^{-3}	2.84×10^{-3}
	u_3 [m]	5.3798×10^{-3}	5.3784×10^{-3}
	θ_3 [°]	0.1175	0.1157
	M_3 [Nm]	776643	794408.6875
$f=10^3$	time [sec]	11	302
	v_3 [m]	2.4558×10^{-2}	2.3061×10^{-2}
	u_3 [m]	5.3804×10^{-2}	5.4085×10^{-2}
	θ_3 [°]	1.328	1.3508
	M_3 [Nm]	6.8356×10^6	6.94×10^6

Table 3: Calculation time results, displacements and moments in node 3 for case 1 of loads.

Case 2: $\boldsymbol{P}_3 = 0$ N, $\boldsymbol{P}_4 = 5 \times 10^5$ [N], $\boldsymbol{q}_2 = 2 \times 10^3$ [N/m], $\boldsymbol{q}_3 = 4 \times 10^4$ [N/m], $\boldsymbol{q}_4 = 0$ and $\boldsymbol{q}_{10} = 5 \times 10^4$ [N/m].

		Power series	FEM
$f = 1$	time [sec]	9	22
	v_3 [m]	3.792×10^{-5}	3.1×10^{-5}
	u_3 [m]	2.88×10^{-5}	2.9×10^{-5}
	θ_3 [°]	5.459×10^{-3}	5.531×10^{-3}
	M_3 [Nm]	72695.8	73705.85
$f = 10^2$	time [sec]	9	144
	v_3 [m]	7.016×10^{-3}	7.418×10^{-3}
	u_3 [m]	2.843×10^{-3}	2.902×10^{-3}
	θ_3 [°]	0.568	0.562
	M_3 [Nm]	7.07×10^6	7.402×10^6

Table 4: Calculation time results, displacements and moments in node 3 for case 2 of loads.

It is observed from these tables that in general the values are of the same order, taking into account that the processes are totally different to arrive at them. In this plane frame, 20 elements per bar have been used in finite elements, which makes a total of 600 unknowns. Using power series, they are only 15 unknowns. The information referring to the computational calculation time has also been added, observing a very large difference between the two methods, especially in the presence of loads that cause large deformations.

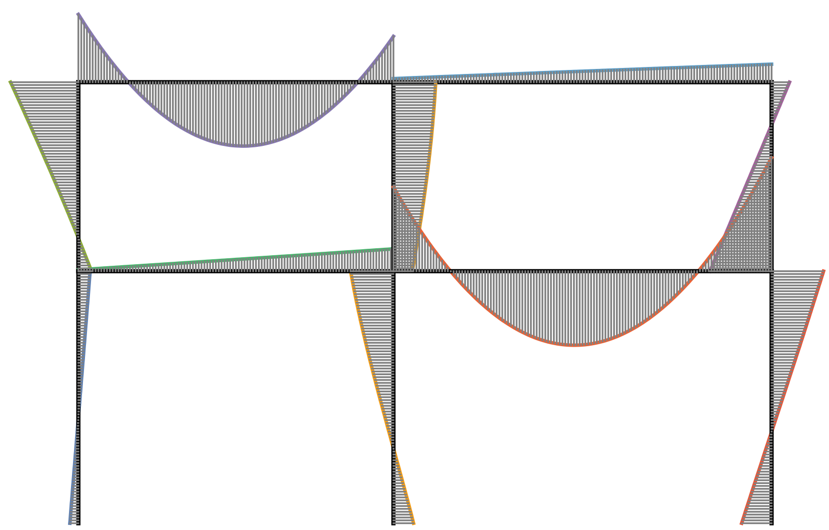


Figure 9 - Moment diagram for Case 2 with f = 10.

Something that can be clearly reflected in this example is the very low computational time consumed using series of powers. Each situation has been solved in both programs using the same computer and under the same conditions. As the load state increases, the large deformations increase and the computational time increases enormously when using finite elements, the same does not happen if power series are used. This concludes in another of the great advantages of the proposed method.

CONCLUSIONS

The most relevant feature to be highlighted in the solution by power series to obtain the natural frequencies of a plane frame is the number of unknowns. It is always very small the amount of unknowns that solves the structural model, whatever the number of bars of the plane frame. In other methods one must necessarily pose three unknowns for each node. In general, it is necessary to subdivide each bar into small elements to guarantee a better result, which enforces the proliferation of nodes, increasing the number of unknowns of the problem to be solved.

The solutions are completely based on the use of infinite series of integer powers. This form of infinite series, to be applied for practical purposes, must be truncated. Nevertheless, the exact solution (or with arbitrary precision) can be achieved, when the series' degree tends to infinity in the limit and the response (numerical or in closed form) converges monotonically. The accuracy of the results is subject to the maximum power of the adopted series, so that there is always the possibility of improving the response by increasing the number of terms of the series, and thus achieving the required precision. When using series, the first natural frequencies require low power exponents, with powers up to order 10 there is an excellent precision. For higher frequencies, it is necessary to increase the degree in the polynomials, although with grade 20 also excellent results are obtained without additional difficulties and extra computing time. It should be taken into account that it is also necessary to increase the number of elements when using FEM, greatly increasing the number of unknowns and computing time, on the other hand when using powers series the number of unknowns is independent of the order of the polynomial used.

While it is true that sophisticated programs of the finite element method are available, it is also true that most of them have closed codes that do not allow some modifications to model complex boundary conditions in structural problems. Under this context and given the need to solve more sophisticated frame problems (non-linear boundary conditions, restricted motion in the joints, etc), practitioners often make equivalent models, which leads to dangerously simplified representations. As a specific application in structural engineering, a solution in power series has been presented here by means of a systematic recurrence algorithm. This allows analyzing the structural behavior of plane frames, arriving at an elastic solution taking into account the theory of second order.

A remarkable point in the solution of plane frames problems by means of MPS is the actual number of unknowns. Whatever the number of bars or members of the frame, the number of unknowns solved by the structural assembly is always very small, independently of the polynomial order of the series. In comparison the effective number of unknowns in MPS can be 20 to 100 times lower than in the case of other methods. This is not only reflected in the reduction of data but more importantly in the spare of computational time (between 3 to 10 times faster that other methods for the same problem test). Low time consuming routines are always of importance and extreme necessity in the case of uncertainty quatification, stochastic modelling and optimization procedeures. Further applications of the present methodology would be oriented in the aforementioned problems.

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